ECS 455 Chapter 1

Introduction & Review

1.2 Fourier Transform and Communication System

Office Hours:
BKD 3601-7

Tuesday	**9:30-10:30**
Tuesday	**13:30-14:30**
Thursday	**13:30-14:30**

7 Equations

that changed the world

… and still rule everyday life

$$\nabla \cdot E = 0$$

$$\nabla \times E = -\frac{1}{c}\frac{\partial H}{\partial t}$$

$$\nabla \cdot H = 0$$

$$\nabla \times H = \frac{1}{c}\frac{\partial E}{\partial t}$$

$$i\hbar \frac{\partial}{\partial t}\psi = \hat{H}\psi$$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

First among equals

Behind the scenes, equations rule our everyday lives. Mathematician Ian Stewart goes in search of the most influential

THE alarm rings. You glance at the clock. The time is 6.30 am. You haven't even got out of bed, and already at least six mathematical equations have influenced your life. The memory chip that stores the time in your clock couldn't have been devised without a key equation in quantum mechanics. Its time was set by a radio signal that we would never have dreamed of inventing were it not for James Clerk Maxwell's four equations of electromagnetism. And the signal itself travels according to what is known as the wave equation.

We are afloat on a hidden ocean of equations. They are at work in transport, the financial system, health and crime prevention and detection, communications, food, water, heating and lighting. Step into the shower and you benefit from equations used to regulate the water supply. Your breakfast cereal comes from crops that were bred with the help of statistical equations. Drive to work and your car's aerodynamic design is in part down to the Navier-Stokes equations that describe how air flows over and around it. Switching on its satnav involves quantum physics again, plus Newton's laws of motion and gravity, which helped launch the geopositioning satellites and set their orbits. It also uses random number generator equations for timing signals, trigonometric equations to compute location, and special and general relativity for precise tracking of the satellites' motion under the Earth's gravity.

Without equations, most of our technology would never have been invented. Of course, important inventions such as fire and the wheel came about without any mathematical knowledge. Yet without equations we would be stuck in a medieval world.

Equations reach far beyond technology too. Without them, we would have no understanding of the physics that governs the tides, waves breaking on the beach, the ever-changing weather, the movements of the planets, the nuclear furnaces of the stars, the spirals of galaxies – the vastness of the universe and our place within it.

There are thousands of important equations. The seven I focus on here – the wave equation, Maxwell's four equations, the Fourier transform and Schrödinger's equation – illustrate how empirical observations have led to equations that we use both in science and in everyday life.

First, the wave equation. We live in a world of waves. Our ears detect waves of compression in the air as sound, and our eyes detect light waves. When an earthquake hits a town, the destruction is caused by seismic waves moving through the Earth.

Mathematicians and scientists could ›

Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Complex exponential

$$\cos(A) = \mathrm{Re}\{e^{jA}\} = \frac{1}{2}\left(e^{jA} + e^{-jA}\right)$$

$$\sin(A) = \mathrm{Im}\{e^{jA}\} = \frac{1}{2j}\left(e^{jA} - e^{-jA}\right)$$

$$\cos(-x) = \cos(x)$$

$$\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$$

$$2\cos^2 x = 1 + \cos(2x)$$

$$2\sin^2 x = 1 - \cos(2x)$$

$$2\sin(x)\cos(x) = \sin(2x)$$

$$\frac{d}{dx}\sin x = \cos x$$

$$\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$$

(product-to-sum formula)

(Continuous-Time) Fourier Transform

Time Domain | Frequency Domain

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df \overset{\mathcal{F}}{\rightleftharpoons} G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt$$

Complex exponential: $e^{j2\pi ft} = \cos(2\pi f)t + j\sin(2\pi ft)$

$$g(0) = \int_{-\infty}^{\infty} G(f) df$$

$$G(0) = \int_{-\infty}^{\infty} g(t) dt$$

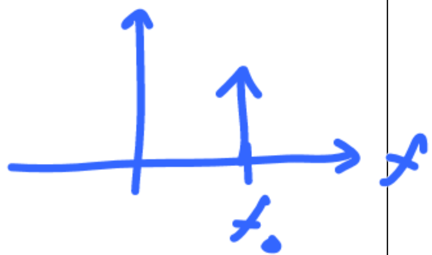
Fourier Transform Pairs (1)

Time Domain | Frequency Domain

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df \overset{\mathcal{F}}{\rightleftharpoons} G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt$$

$\cos(2\pi f_0 t) + j\sin(2\pi f_0 t) =$

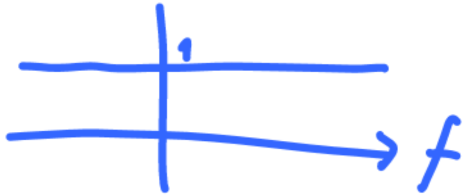
$$e^{j2\pi f_0 t} \overset{\mathcal{F}}{\rightleftharpoons} \delta(f - f_0)$$



$f_0 = 0$

$$1 \overset{\mathcal{F}}{\rightleftharpoons} \delta(f)$$

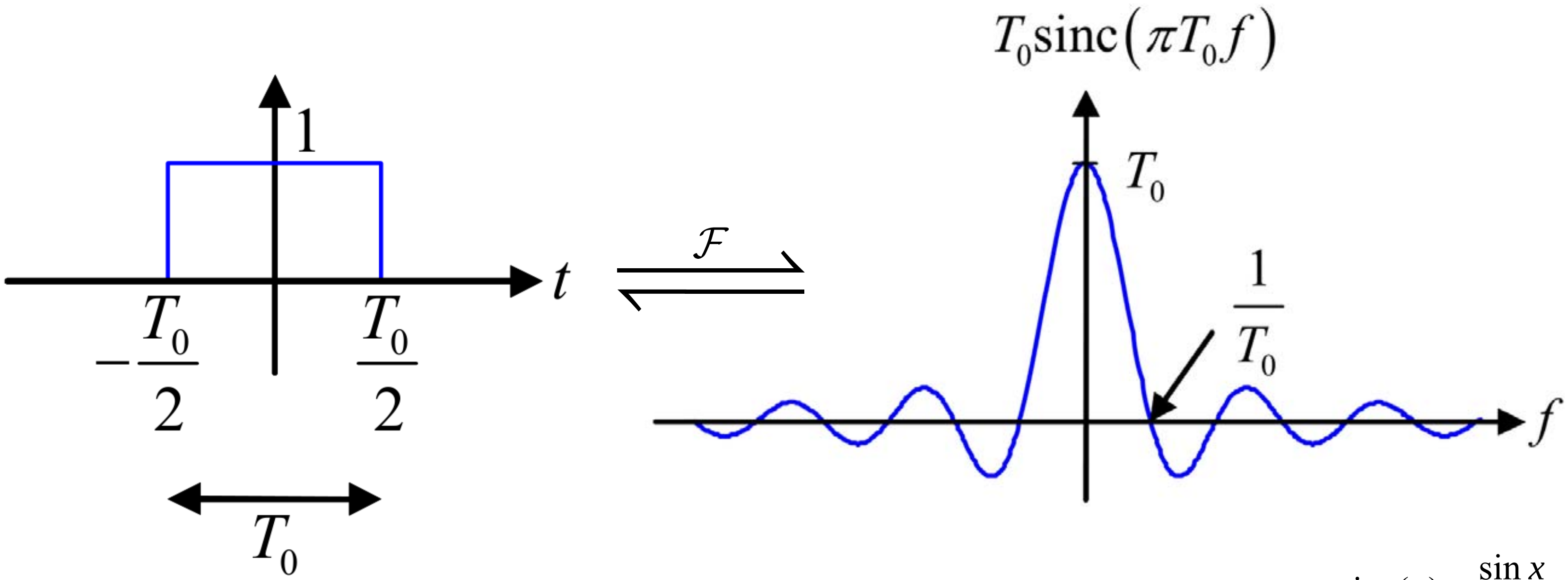
$$\delta(t) \overset{\mathcal{F}}{\rightleftharpoons} 1$$



Fourier Transform Pairs (2)

Time Domain | Frequency Domain

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df \overset{\mathcal{F}}{\rightleftharpoons} G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt$$

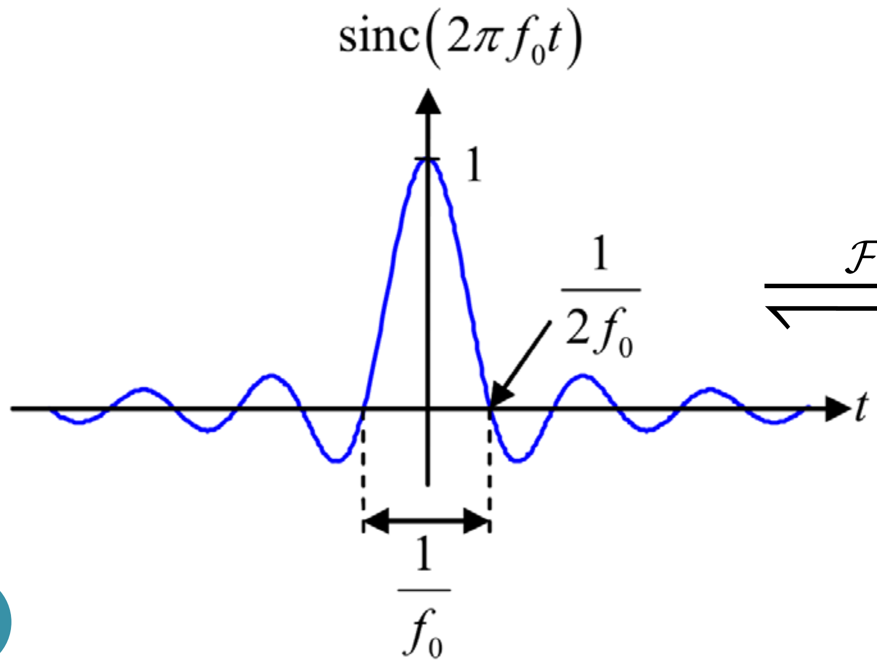


$\text{sinc}(x) = \frac{\sin x}{x}$

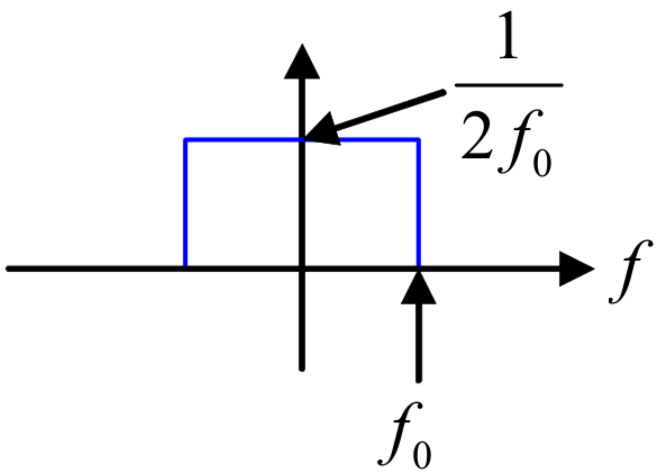
Fourier Transform Pairs (3)

Time Domain | Frequency Domain

$$g(t)=\int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df \overset{\mathcal{F}}{\rightleftharpoons} G(f)=\int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt$$



$\overset{\mathcal{F}}{\rightleftharpoons}$



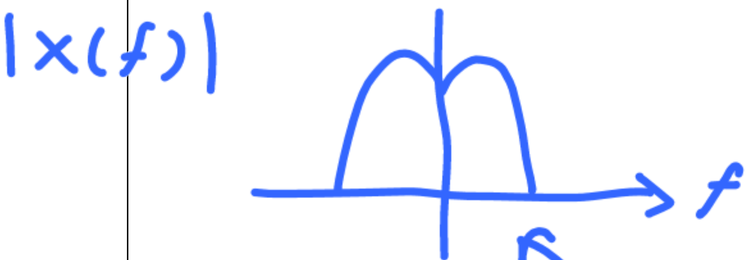
Important Properties

$$\{x * y\}(t) = \int_{-\infty}^{\infty} x(\mu) y(t-\mu) d\mu = \int_{-\infty}^{\infty} x(t-\mu) y(\mu) d\mu$$

Convolution Properties:

$$x * y \overset{\mathcal{F}}{\rightleftharpoons} X \times Y$$

$$x \times y \overset{\mathcal{F}}{\rightleftharpoons} X * Y$$



Shifting Properties:

$$g(t - t_0) \overset{\mathcal{F}}{\rightleftharpoons} e^{-j2\pi f t_0} G(f)$$

Note that the magnitude of this is simply $|G(f)|$

$$e^{j2\pi f_0 t} g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(f - f_0)$$

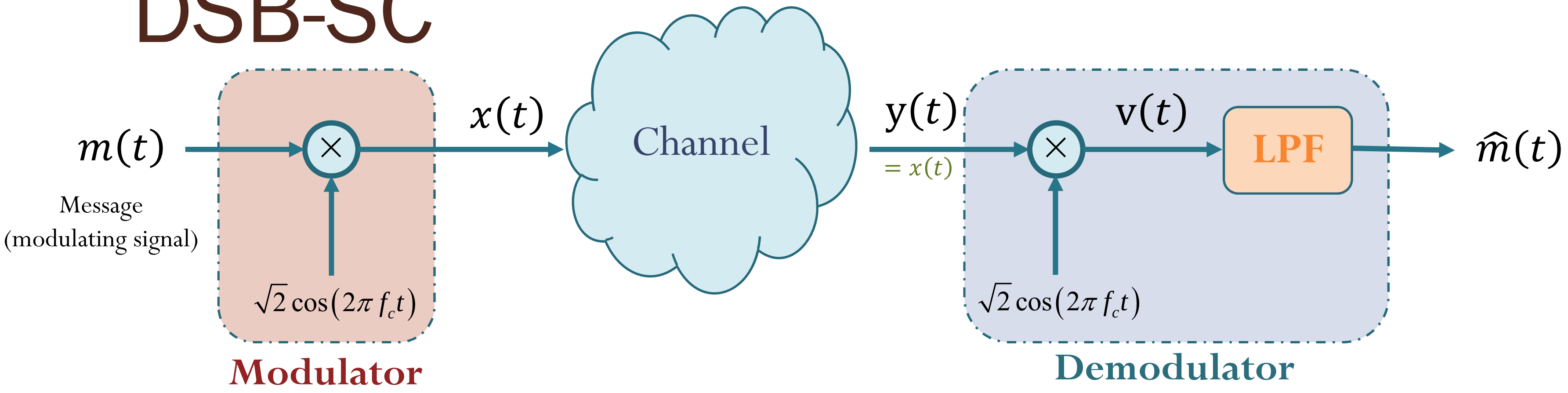
y(t) = x(t-5)

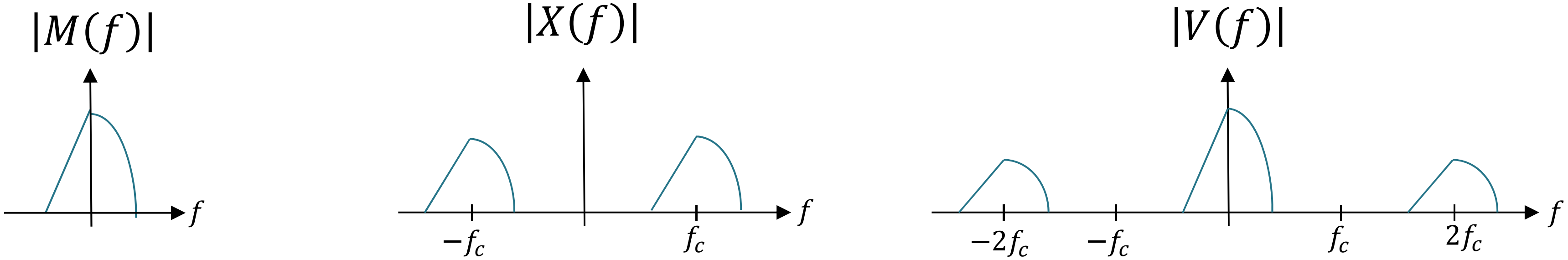
|Y(f)| =

Modulation:

$$g(t)\cos(2\pi f_c t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2} G(f - f_c) + \frac{1}{2} G(f + f_c)$$

DSB-SC



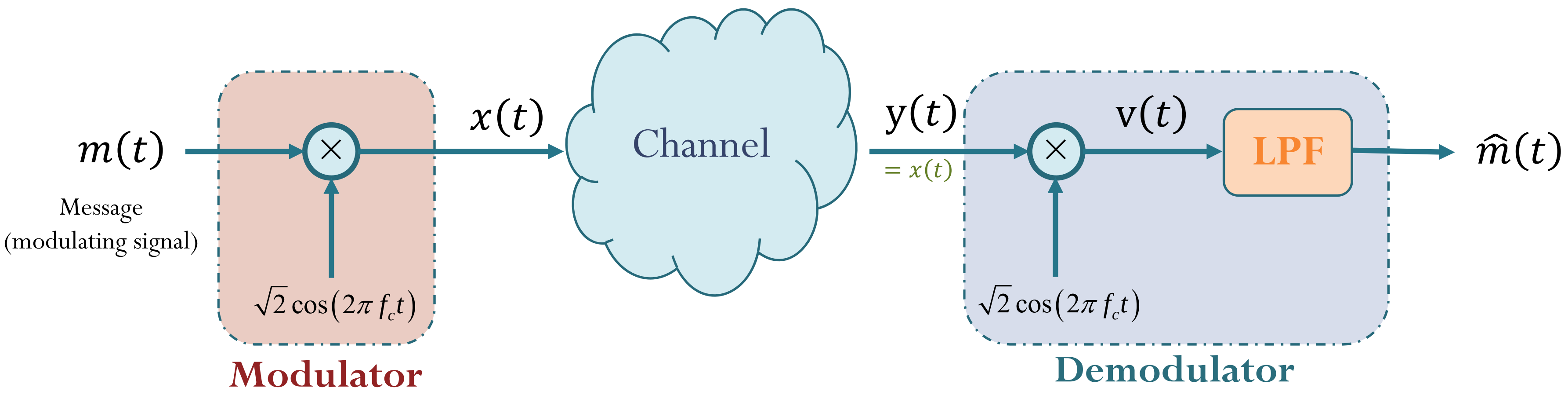


$$g(t)\cos(2\pi f_c t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2}G(f - f_c) + \frac{1}{2}G(f + f_c)$$

$$m(t)\times\left(\sqrt{2}\cos(2\pi f_c t)\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{\sqrt{2}}{2}M(f - f_c) + \frac{\sqrt{2}}{2}M(f + f_c)$$

$$= \frac{1}{\sqrt{2}}M(f - f_c) + \frac{1}{\sqrt{2}}M(f + f_c)$$

DSB-SC

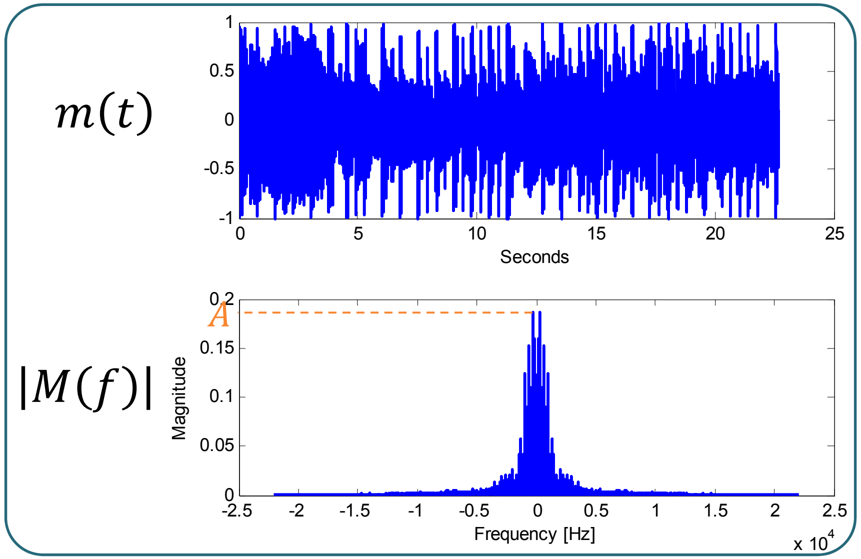


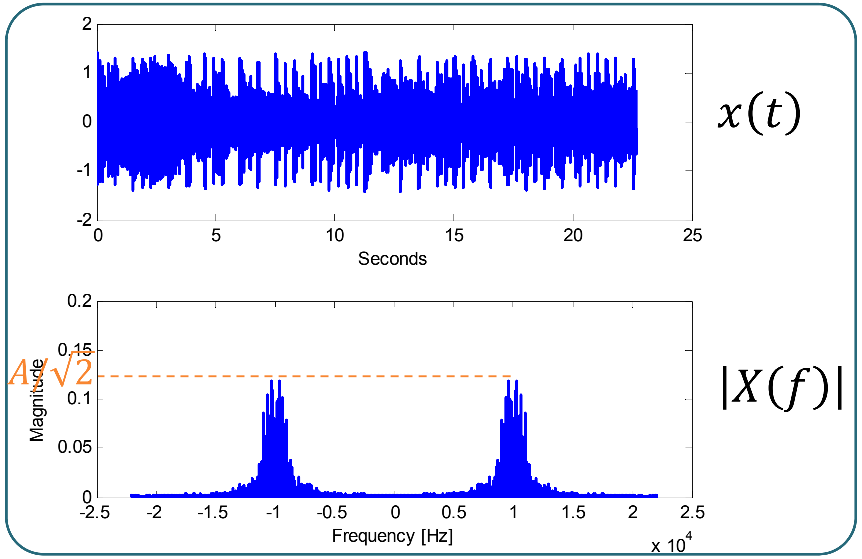
Key equation:

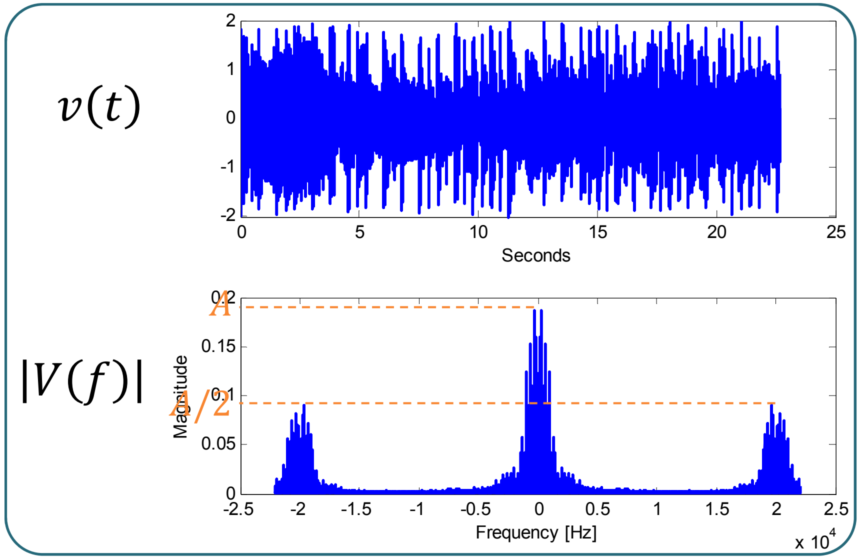
$$\text{LPF}\left\{ \underbrace{\underbrace{\left(m(t)\times\sqrt{2}\cos(2\pi f_c t)\right)}_{x(t)}\times\left(\sqrt{2}\cos(2\pi f_c t)\right)}_{v(t)} \right\} = m(t)$$

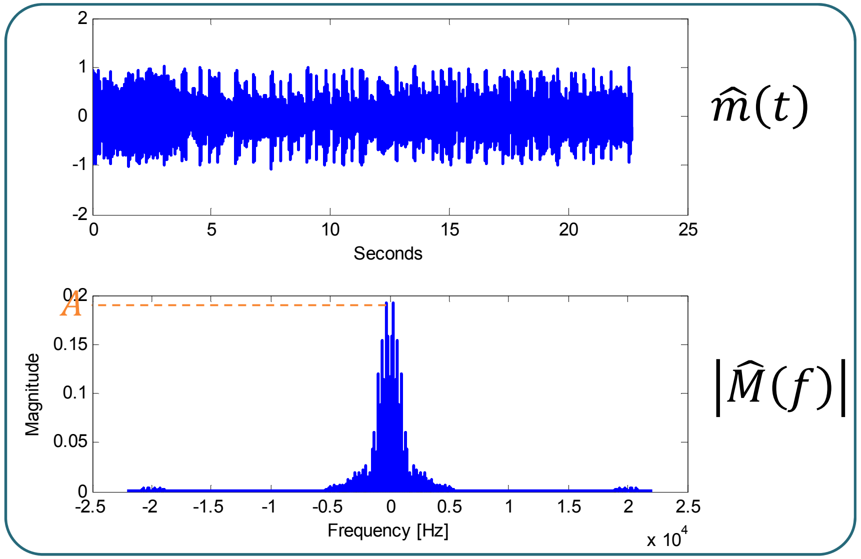
[Demo_DSBSC_Sound_ReadWAV.m]

DSB-SC



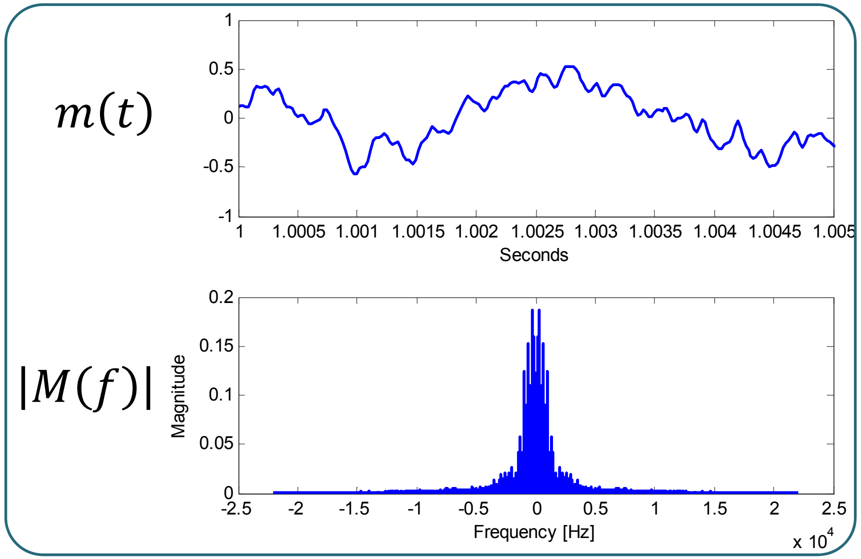


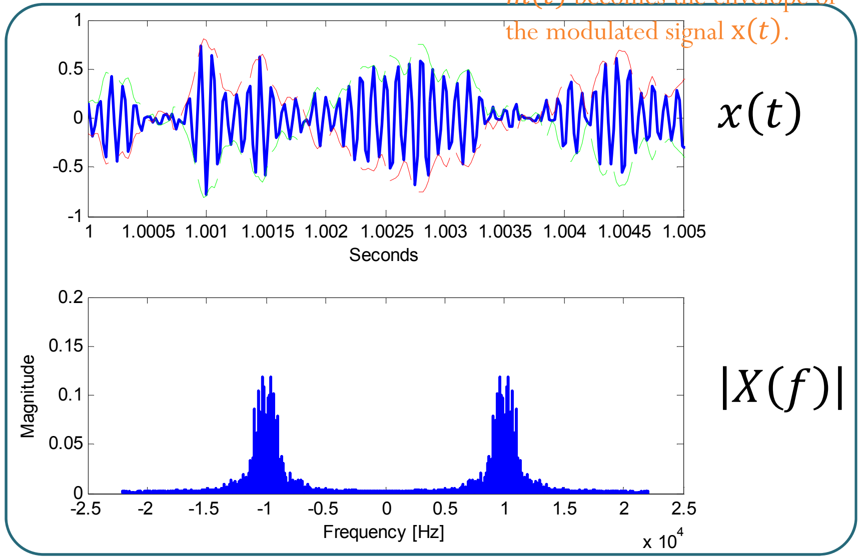


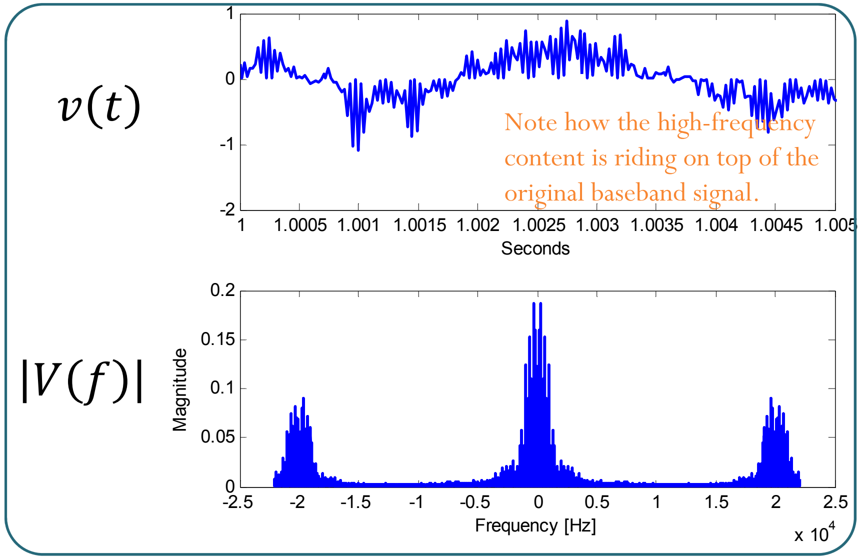


DSB-SC (Zoomed in time)

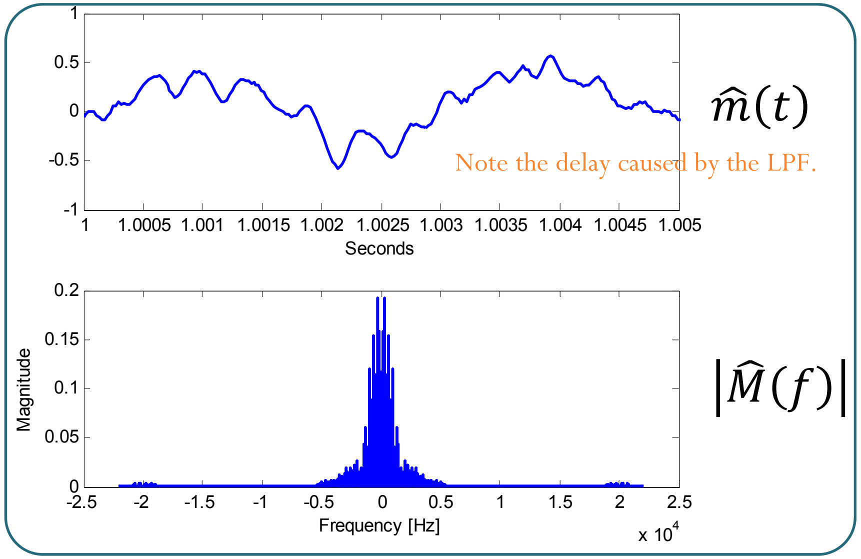
Note how the baseband signal $m(t)$ becomes the envelope of the modulated signal $x(t)$.





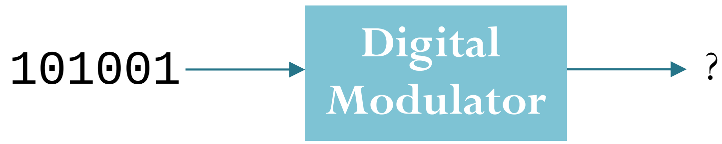


Note how the high-frequency content is riding on top of the original baseband signal.

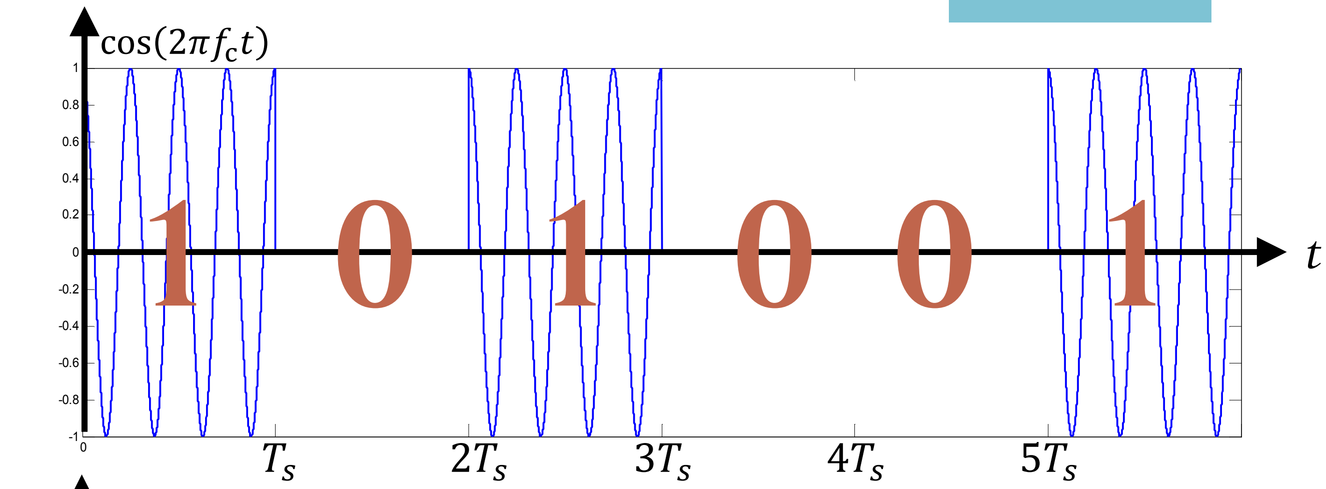


Note the delay caused by the LPF.

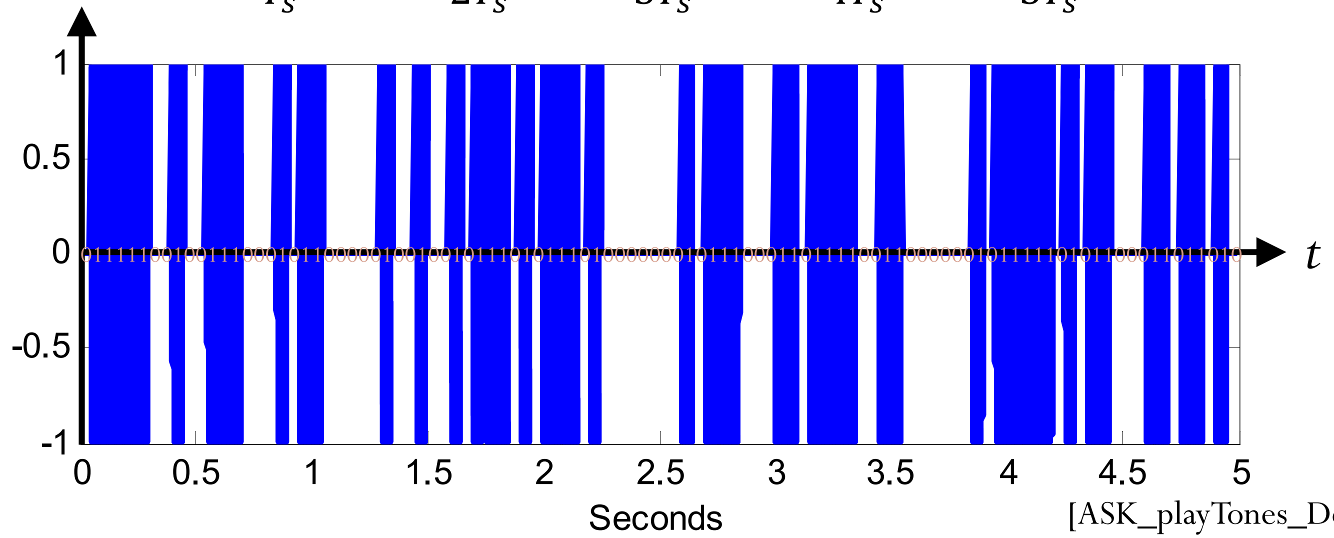
ON-OFF Keying (OOK)



$f_c = 4$ Hz
Bit rate = 1 bps



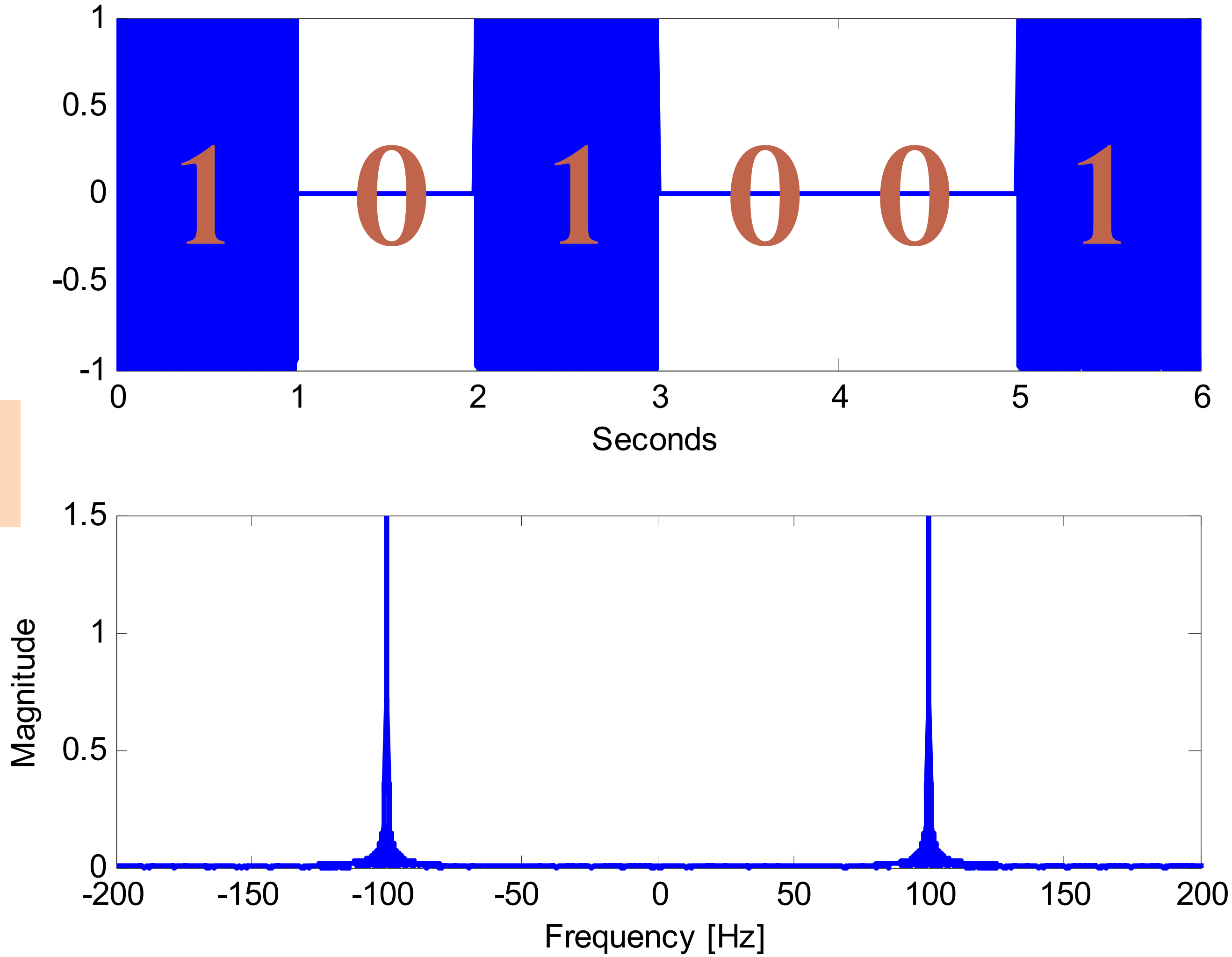
$f_c = 100$ Hz
Bit rate = 20 bps



[ASK_playTones_Demo.m]

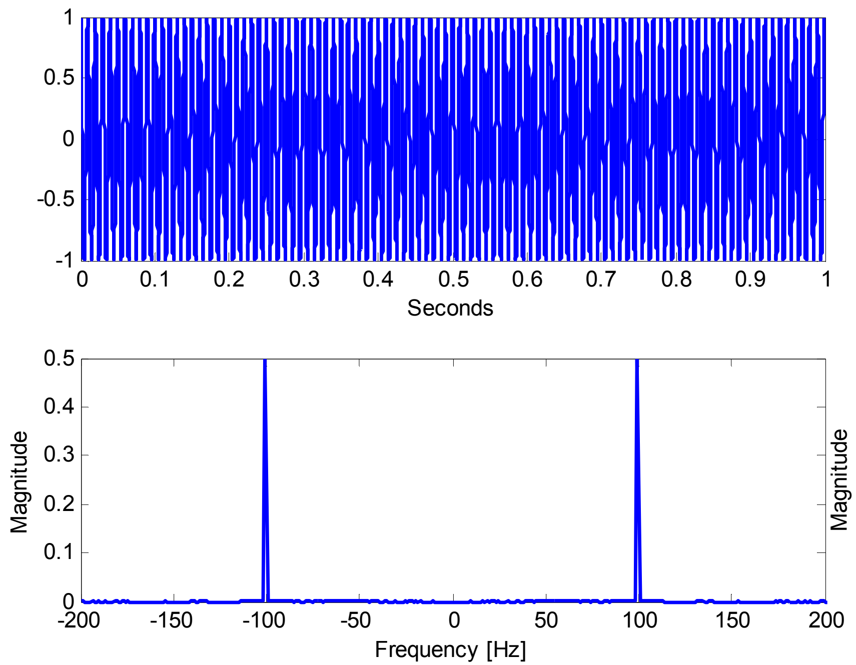
Spectrum of ON-OFF Keying

$f_c = 100$ Hz
Bit rate = 1 bps

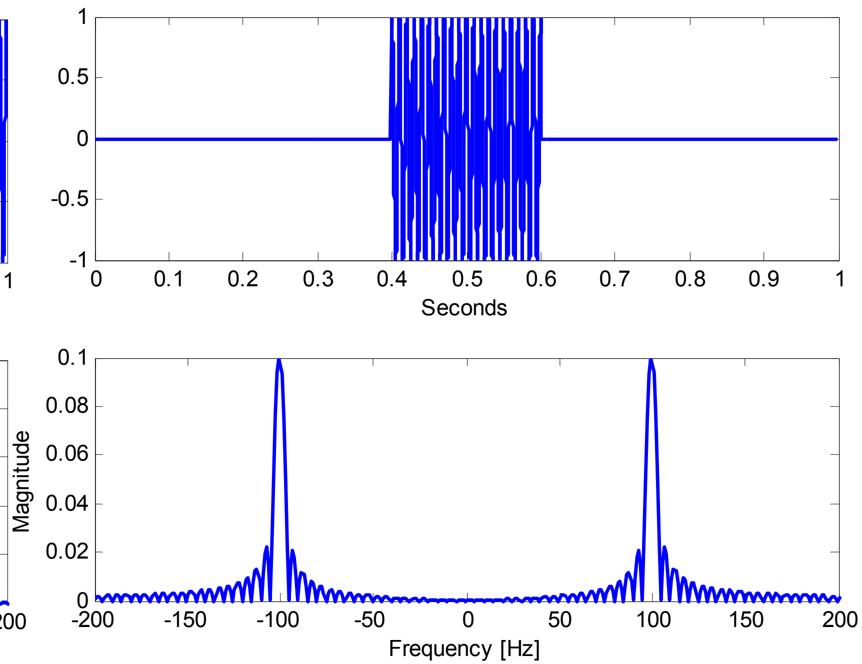


Cos vs. Cos Pulse

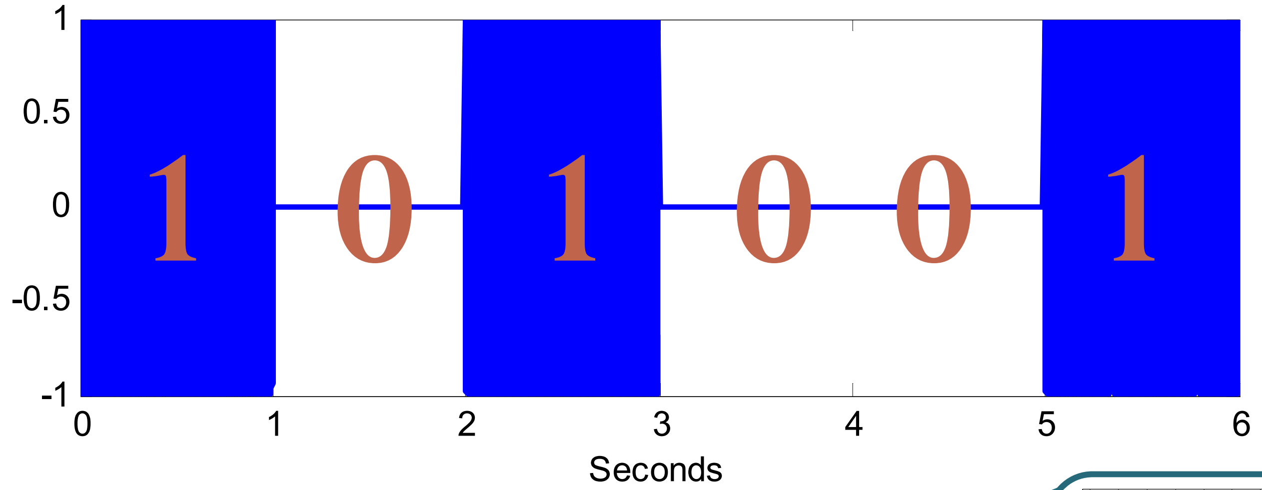
$$x(t) = \cos(2\pi(100)t)$$



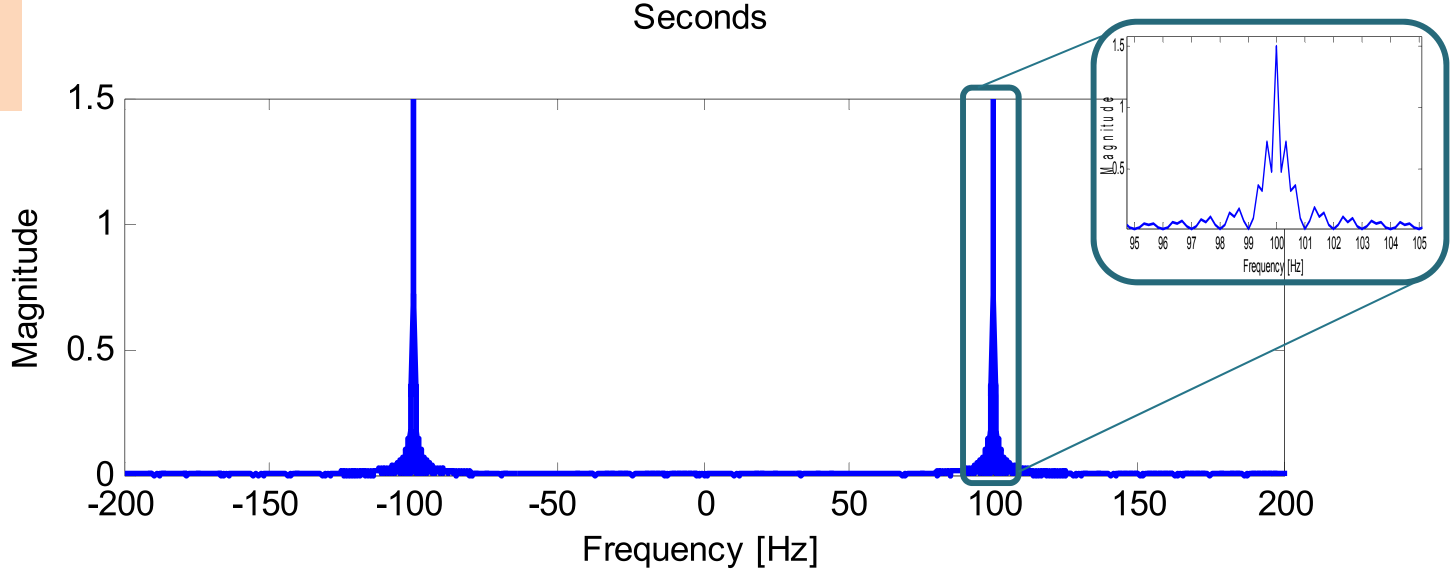
$$x(t) = \begin{cases} \cos(2\pi(100)t), & 0.4 \le t \le 0.6, \\ 0, & \textit{otherwise}. \end{cases}$$



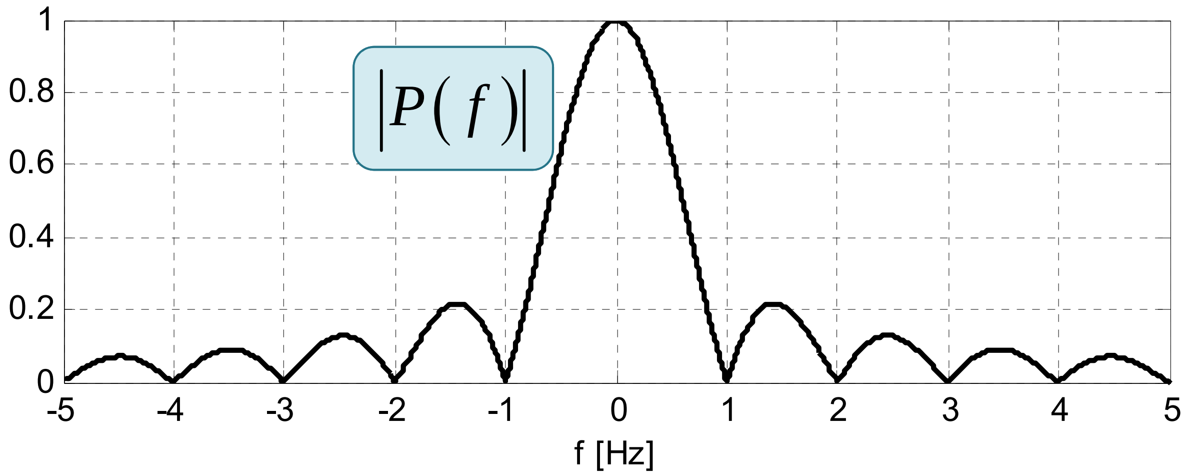
Spectrum of ON-OFF Keying

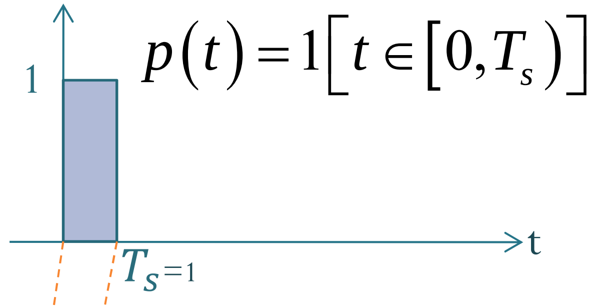


$f_c = 100$ Hz
Bit rate = 1 bps

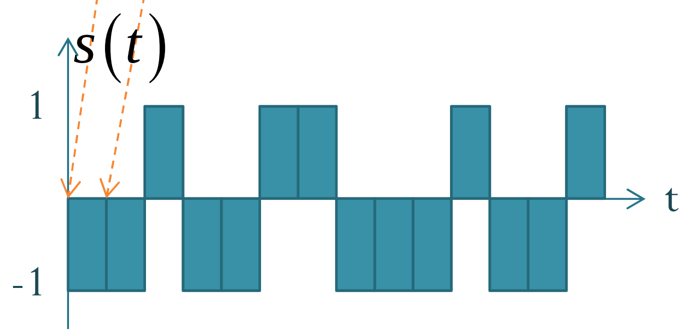


Spectrum of a train of rect. pulses (1/4)





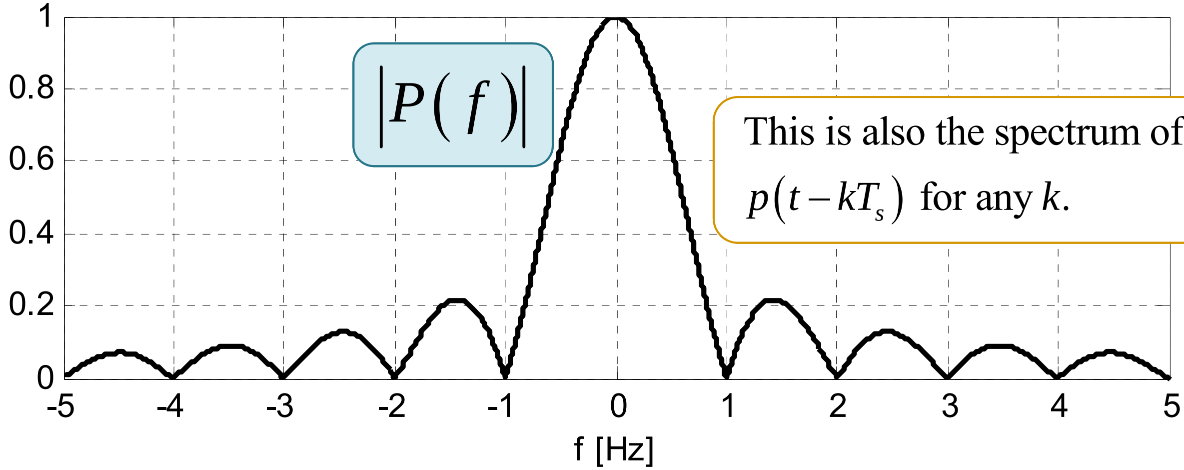
A= [-1,-1,1,-1,-1,1,1,-1,-1,-1,1,-1,-1,1,-1,1,1,-1,-1,-1,-1,1,-1,-1,-1,-1,-1,1,-1,1]



$$s(t) = \sum_{k=0}^{n-1} A_k \, p(t - kT_s)$$

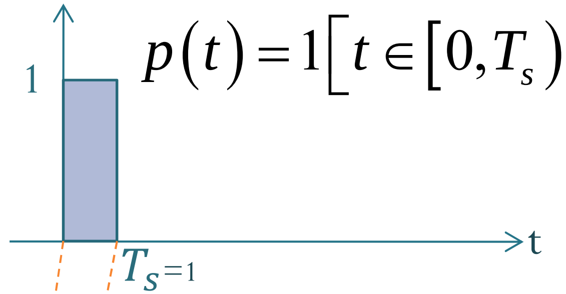
Can you sketch the spectrum of $s(t)$?

Spectrum of a train of rect. pulses (2/4)

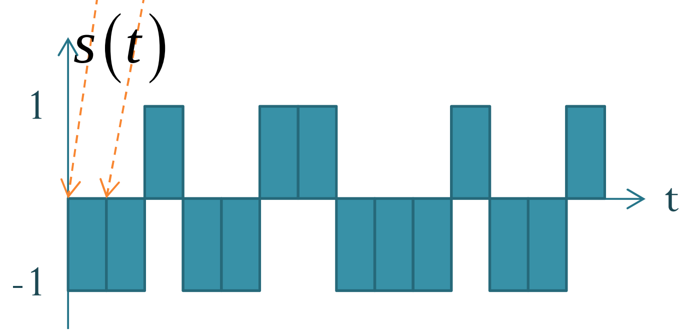


This is also the spectrum of $p(t-kT_s)$ for any k.

$$p(t) = 1\left[t \in [0, T_s)\right]$$



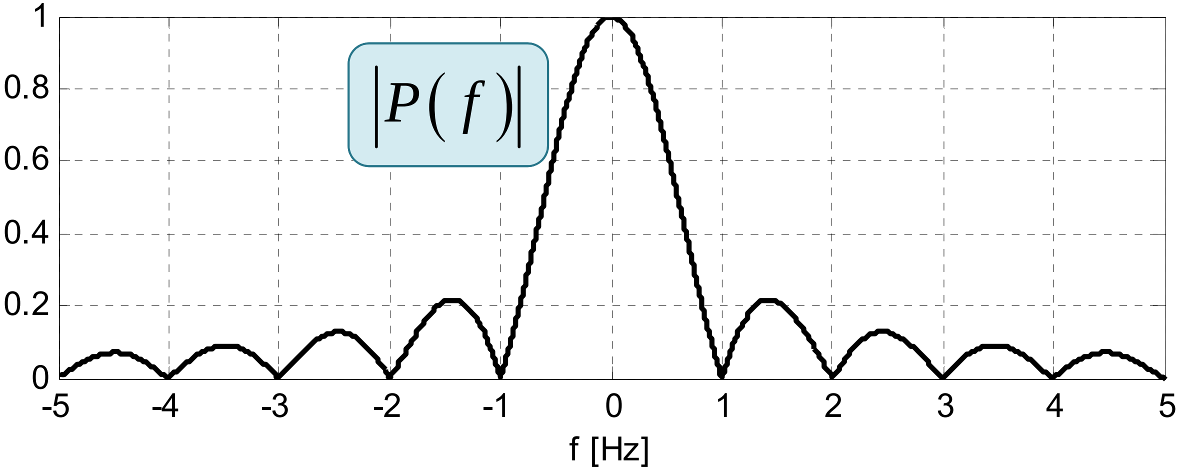
A= [-1,-1,1,-1,-1,1,1,-1,-1,-1,1,-1,-1,1,-1,1,1,-1,-1,-1,-1,1,-1,-1,-1,-1,-1,1,-1,1]

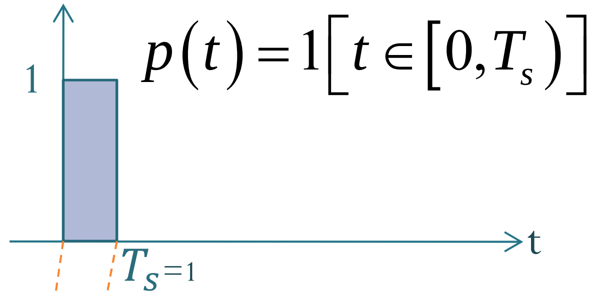


$$s(t) = \sum_{k=0}^{n-1} A_k \, p(t-kT_s)$$

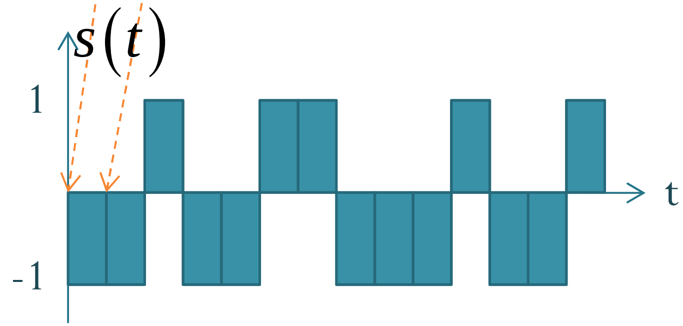
Does this mean $|S(f)|$ will simply be a sum of $|P(f)|$ and therefore its shape will be similar to $|P(f)|$?

Spectrum of a train of rect. pulses (3/4)





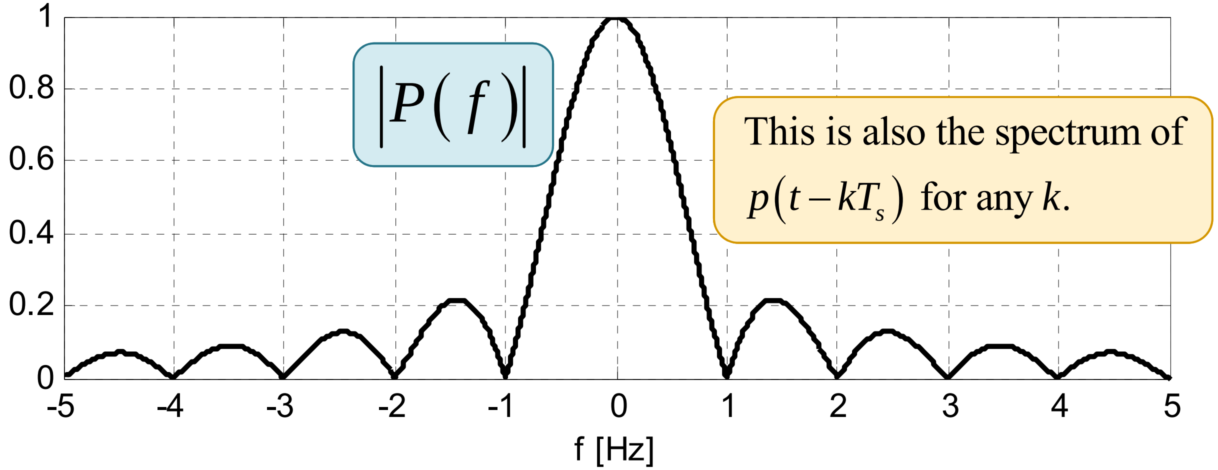
$A=$ [-1,-1,1,-1,-1,1,1,-1,-1,-1,1,-1,-1,1,-1,1,1,-1,-1,-1,-1,1,-1,-1,-1,-1,-1,1,-1,1]

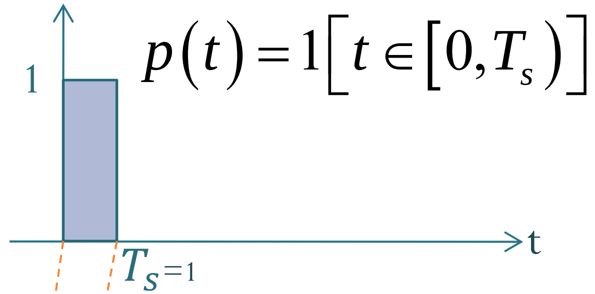


$$s(t)=\sum_{k=0}^{n-1}A_k\,p(t-kT_s)$$

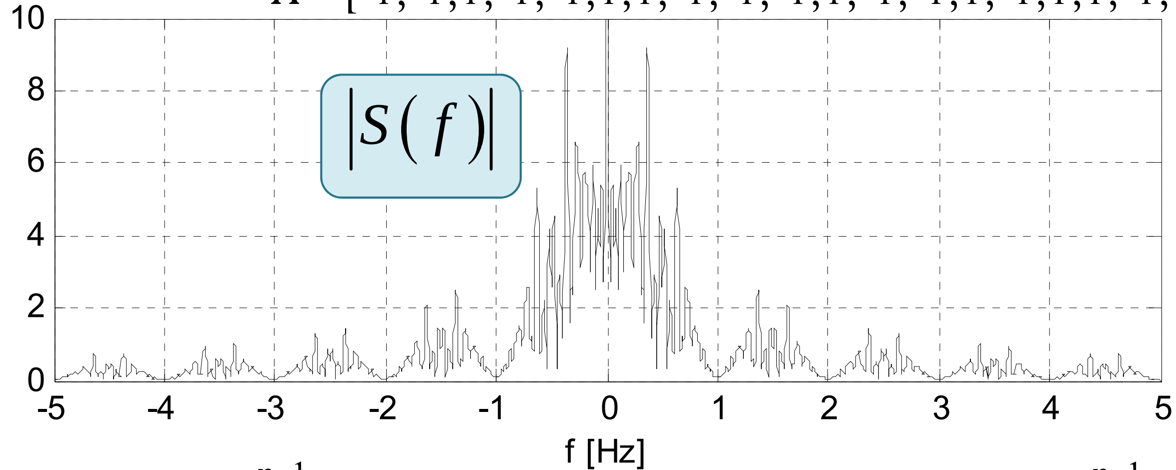
$$\xrightarrow{\mathcal{F}} S(f)=P(f)\sum_{k=0}^{n-1}A_k e^{-j2\pi fkT_s}$$

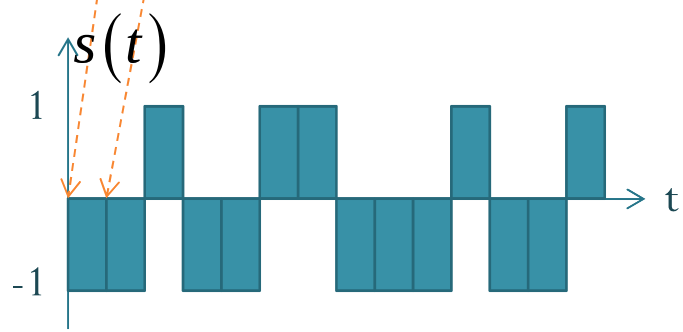
Spectrum of a train of rect. pulses (4/4)





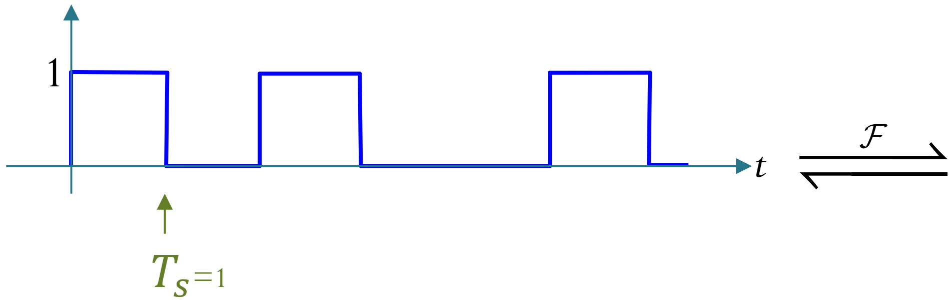
$A=$ [-1,-1,1,-1,-1,1,1,-1,-1,-1,1,-1,-1,1,-1,1,1,-1,-1,-1,-1,1,-1,-1,-1,-1,-1,1,-1,1]

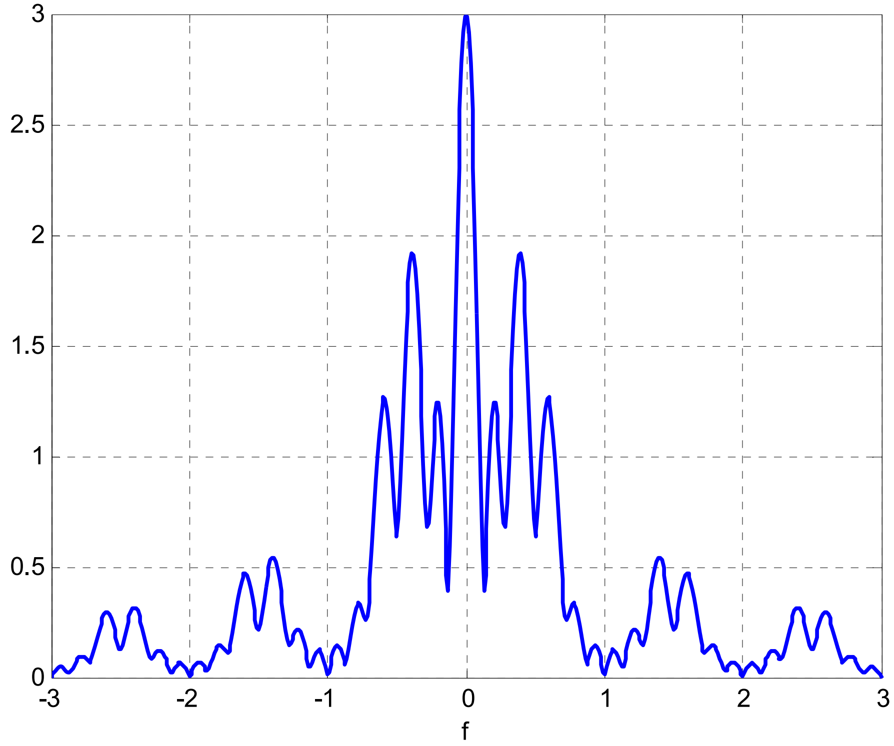




$$s(t)=\sum_{k=0}^{n-1} A_k c(t-kT_s) \xrightarrow{\mathcal{F}} S(f)=P(f)\sum_{k=0}^{n-1} A_k e^{-j2\pi fkT_s}$$

Spectrum of a train of rect. pulses: a revisit to an earlier OOK Example





Spectrum of ON-OFF Keying

$f_c = 100$ Hz
Bit rate = 1 bps